

Lecture 9

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7.2: Trigonometric Integrals

In today's lecture, we will cover 3 types of integrals:

$$\textcircled{1} \int \sin^m x \cos^n x \, dx$$

$$\textcircled{2} \int \sec^m x \tan^n x \, dx$$

$$\textcircled{3} \textcircled{i} \int \sin(mx) \cos(nx) \, dx$$

$$\textcircled{ii} \int \sin(mx) \sin(nx) \, dx$$

$$\textcircled{iii} \int \cos(mx) \cos(nx) \, dx$$

\textcircled{i} This splits into three cases

n odd If n is odd, we can write $n=2p+1$,

then

$$\sin^m x \cos^n x = \sin^m x \cos^{2p+1} x = \sin^m x (\cos^2 x)^p \cos x$$

$$= \sin^m x (1 - \sin^2 x)^p \cos x$$

So, the integral

$$\int \sin^m x \cos^n x \, dx = \int \sin^m x (1 - \sin^2 x)^p \cos x \, dx$$

can be done with a u -substitution of $m-2$

$$u = \quad du =$$

Ex: Compute $\int \sin^2 x \cos^3 x \, dx$

m odd This works just like the last case, but we instead use $\sin^2 x = 1 - \cos^2 x$, and the u -sub

$$u = \quad du =$$

Ex: Compute $\int \sin^5 x \cos^3 x \, dx$

If both m & n are even, we use the identities ¹⁴⁻³

$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \& \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Sometimes, using

$$\sin x \cos x = \frac{1}{2} \sin 2x$$

can be useful. Another alternative is to change everything to powers of sine or cosine using

$$\sin^2 x + \cos^2 x = 1$$

Ex: Compute $\int \sin^4 x \cos^2 x \, dx$

(2) There is again three subcases

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m even & ($n \geq 1$ or $m \geq 4$)

We reserve one factor of $\sec^2 x$ and turn everything else into powers of $\tan x$ using

$$\sec^2 x = 1 + \tan^2 x$$

then we use the u -substitution

$$u = \quad \quad \quad du =$$

Ex: Compute $\int \tan^5 \theta \sec^4 \theta d\theta$

n odd & $m \geq 1$

In this case we reserve a factor of $\tan x \sec x$ and write everything else in terms of $\sec x$.

(this is why we need $m \geq 1$ & taking one power of $\tan x$ leaves an even power of $\tan x$ left). We then use the

$$u\text{-sub: } u = \quad \quad \quad du =$$

Ex: Compute $\int \sec^3 y \tan y \, dy$

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m odd & n even Here we turn all of the tangents into secants (which we can do since there are an even number of them). Then we use one of the formulas for integrals of powers of secant:

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec^3 x \, dx = \frac{\sec x \tan x}{2} + \frac{1}{2} \ln|\sec x + \tan x| + C$$

$$(n \geq 3) \int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

The other integral to know is

$$\int \tan x \, dx = \ln|\sec x| + C$$

Ex: Compute $\int \sec y \tan^2 y \, dy$

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③ To compute the integrals of this type we use the identities

$$\textcircled{1} \sin[(m-n)x] = \sin(mx)\cos(nx) - \cos(mx)\sin(nx)$$

$$\textcircled{2} \sin[(m+n)x] = \sin(mx)\cos(nx) + \cos(mx)\sin(nx)$$

$$\textcircled{3} \cos[(m-n)x] = \cos(mx)\cos(nx) + \sin(mx)\sin(nx)$$

$$\textcircled{4} \cos[(m+n)x] = \cos(mx)\cos(nx) - \sin(mx)\sin(nx)$$

to get the identities

$$\cos(mx)\sin(nx) = \frac{1}{2} [\sin[(m-n)x] - \sin[(m+n)x]] \quad \frac{\textcircled{1} + \textcircled{2}}{2}$$

$$\sin(mx)\sin(nx) = \frac{1}{2} [\cos[(m-n)x] - \cos[(m+n)x]] \quad \frac{\textcircled{3} - \textcircled{4}}{2}$$

$$\cos(mx)\cos(nx) = \frac{1}{2} [\cos[(m-n)x] + \cos[(m+n)x]] \quad \frac{\textcircled{3} + \textcircled{4}}{2}$$

Ex: Compute $\int \cos \pi x \cos 4\pi x dx$

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Now, we work through integrals of powers of secant and tangent.

Ex: Compute $\int \sec x dx$

Ex: Compute $\int \tan x dx$

Ex: $\int \sec^2 x \, dx$

Ex: $\int \tan^2 x \, dx$

Ex: $\int \sec^3 x \, dx$

Ex: $(n \geq 2) \int \tan^n x \, dx$

Ex: ($n \geq 3$) $\int \sec^n x \, dx$